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# Harmonic Problems: Fourier Series and Boundary Element Methods

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**José M. de Aguiar**
Mechanics and Computation - Mechanical Engineering Department
Stanford University , Stanford, California, 94305-3030, USA
**João B. de Aguiar**
Departamento de Mecânica e Fabricação - Escola Politécnica
Escola Politécnia - Universidade de São Paulo

**Abstract.** *The problem of a circular cylindrical shell intercepted by an inclined plane is analyzed for the harmonic operator. The solution is developed using the Fourier series expansion for arbitrary boundary conditions, prescribed along the boundary curve. In addition, an alternative solution is developed using the boundary element method.*
*We have some numerical comparisons concerning the convergence of the both solutions. Some of the weakness of both methods are exposed as the angle of intersection between the inclined plane and the cylinder is increased.*
*It should be pointed out that the development for the harmonic operator is the model for higher-order operators which appear for the cylindrical shells. Consequently, the numerical results for the harmonic operator should show the limitations of the methods when used for higher-order operators.*
**Key Words:** *Shell, Harmonic, Boundary Element, Intersection, Series*

## 1. BASIC DIFFERENTIAL EQUATION

Let us consider the Laplace equation on a cylindrical surface. Let $u(x,\theta)$ be a harmonic function such that:

$$\nabla^2 u = 0 \tag{1}$$

with the boundary conditions

$$\begin{aligned} u(x,\theta) = &\ u^*(x,\theta) && (x = a\cos\theta, \theta) \in \Gamma^1 \\ u(x,\theta) = &\ 0 && (x,\theta) \in \Gamma^2, x \to \infty \end{aligned} \tag{2}$$

$u^*(x,\theta)$ being the prescribed value of function $u$ along the boundary curve. The surface coordinates of the cylindrical shell are $\{x,\theta\}$.

The problem is represented schematically in Figure 1. The situation corresponds to an inclined plane intersecting a cylindrical surface of radius $R$ . The parameter $\bar{a} = \frac{a}{R}$ has been chosen to express the intersection along the generator of the cylinder corresponding to $\theta = 0$.

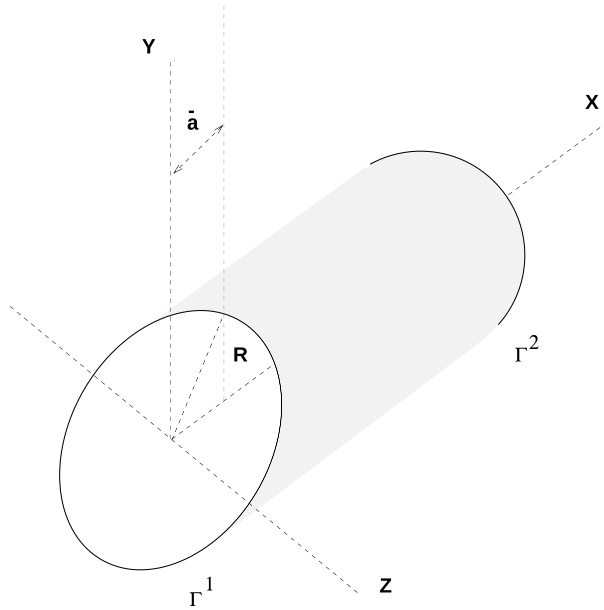


**Figure 1:** Harmonic Problem on a Cylindrical Surface

The function $u(\mathbf{x}, \theta)$ can be assumed to be the temperature field on the cylindrical surface which is developed as a consequence of the prescribed temperature field kept along the boundary curve.

We are looking for the solutions of Laplace's equation related to the conditions prescribed along the boundary curve $\Gamma^1$. Consequently, no internal source distribution is placed on equation 1. The extension for those cases can be obtained by adding a particular solution.

Boundary curve $\Gamma^2$ is a circle on the other extreme of the cylinder. The ends of the cylinder are assumed far enough apart to consider a semi-infinite cylindrical surface.

For the cylindrical surface, Laplace's equation is expressed in the form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

being $r = R$ , a constant value for a cylindrical surface.

We are interested in a subset of solutions which are bounded and periodic since we have a semi-infinite cylinder. Under those conditions, the method of separation of variables give us:

$$u(x, \theta) = \sum_{n=0,1,2,\ldots}^{\infty} [A_n \cos(n\theta) + B_n \sin(n\theta)] e^{-n\frac{x}{R}} \tag{3}$$

Restricting further for only symmetric solutions in $\theta$, we select the subset of solutions:

$$u(x, \theta) = \sum_{n=0,1,2,\ldots}^{\infty} A_n \cos(n\theta) e^{-n\frac{x}{R}} \tag{4}$$

It should be pointed out that the eigenvalues for the harmonic operator are integers as found by equation 3. For the case of higher-order operators we would find complex values of the parameter $\lambda$.

## 2. POTENTIAL BOUNDARY CONDITION

Let us consider the use of dimensionless variables, defined for the cylindrical coordinate system by:

$$\bar{x} = \frac{x}{R}\,;\bar{y} = \frac{y}{R}\,;\bar{z} = \frac{z}{R};\;\bar{a} = \frac{a}{R}$$

The boundary curve can be described in terms of the dimensionless variables by:

$$\begin{cases} \bar{x} = \bar{a}\cos\theta \\ \bar{y} = \cos\theta \\ \bar{z} = \sin\theta \end{cases}$$

Let us consider the expansion of the solution represented by equation 4 along the boundary curve as described above. From Abramowitz 9.6.34 we have:

$$e^{z\cos(\theta)} = I_0(z) + 2\sum_{k=1,2,\dots}^{\infty} I_k(z)\cos(k\theta) \tag{5}$$

where $I_k(z)$ corresponds to the Bessel Function of second kind of order $k$ in the variable $z$[1].

Expanding the exponential function along the boundary curve in terms of Bessel functions and harmonic functions as given by equation 4, we obtain:

$$u(\bar{a}\cos\theta,\theta) = \sum_{n=0,1,\dots}^{\infty} A_n\cos n\theta\{I_0(-n\bar{a}) + 2\sum_{k=1,2,\dots}^{\infty} I_k(-n\bar{a})\cos(k\theta)\} \tag{6}$$

Introducing trigonometrical identities for the cosine expansion into the equation above, we obtain the following expression for the right-hand side of the series solution along the boundary curve:

$$\begin{aligned}\sum_{n=0,1\dots}^{\infty} A_n\{I_0(-n\bar{a})\cos(n\theta) + \dots \\ &[I_1(-n\bar{a})\cos(n-1)\theta + I_1(-n\bar{a})\cos(n+1)\theta] \\ &+\dots \\ &[I_k(-n\bar{a})\cos(n-k)\theta + I_k(-n\bar{a})\cos(n+k)\theta]\}\end{aligned} \tag{7}$$

The expression above can be interpreted as saying that the n-harmonic solution is spread into the $(n-k)$ and $(n+k)$ harmonic contributions being each contribution proportional to $I_k(-n\bar{a})$. In other words, there is a weighting factor relating the relevance of the contributions. Consequently, a given harmonic, say $ih$ will be matched by contributions from the solutions in several different harmonics.

In general the coefficients for the Fourier harmonics include contributions in the following forms:

$$\begin{cases} I_{n-ih}(-n\bar{a}) + I_{n+ih}(-n\bar{a}) & ih < n \\ \\ I_{ih-n}(-n\bar{a}) + I_{n+ih}(-n\bar{a}) & ih \geq n \end{cases}$$

[1]The definition of the Bessel Function in terms of the complex variable z:

$$I_m(z) = (\frac{1}{2}z)^m \sum_{k=0,1,\dots}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(k+m+1)}$$

In a matrix form we can represent the Fourier coefficients of any prescribed function $u(\bar{x},\theta)$ along the boundary curve in terms of the Fourier coefficients of the solution to the Laplace's equation and a coefficient matrix:

$$\left\{ \begin{array}{c} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{array} \right\} = \left[ \begin{array}{ccc} I_0(0) & \dots & I_n(-n) \\ I_1(-0) + I_1(-0) & \dots & I_{n-1}(-n\bar{a}) + I_{n+1}(-n\bar{a}) \\ I_2(-0) + I_2(-0) & \dots & I_{n-2}(-n\bar{a}) + I_{n+2}(-n\bar{a}) \\ I_3(-0) + I_3(-0) & \dots & I_{n-3}(-n\bar{a}) + I_{n+3}(-n\bar{a}) \\ \vdots & \vdots & \vdots \\ I_n(-0) + I_n(-0) & \dots & I_{n-n}(-n\bar{a}) + I_{n+n}(-n\bar{a}) \end{array} \right] \left\{ \begin{array}{c} \mathcal{A}_0 \\ \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \\ \vdots \\ \mathcal{A}_n \end{array} \right\} \tag{8}$$

We represent equation 8 by:

$$\mathbf{u} = \mathbf{I_u}\mathcal{A} \tag{9}$$

being $\mathcal{A}$ the matrix of the Fourier coefficients. It constitutes the matrix form of the harmonic-Bessel series of the potential problem for the prescribed boundary conditions.

## 3. BONDARY CURVE GEOMETRY

Let us consider the geometry of the boundary curve. Figure 2 depicts the important vectors to be taken into account.

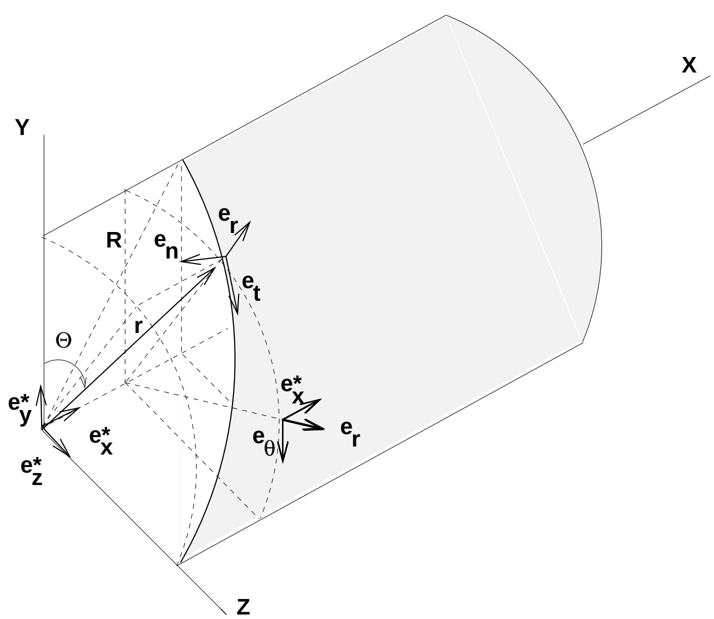


**Figure 2:** Vectors along Boundary Curve

The position vector along the boundary curve is described by:

$$\mathbf{r} = \bar{a}\cos{(\theta)}\,\mathbf{e^*_x} + \cos{(\theta)}\,\mathbf{e^*_y} + \sin{(\theta)}\,\mathbf{e^*_z}$$

where $\{\mathbf{e^*_x}, \mathbf{e^*_y}, \mathbf{e^*_z}\}$ are unit vectors of the global system of reference $\{X, Y, Z\}$.

The vector tangent to the boundary curve corresponds to:

$$\frac{d\mathbf{r}}{d\theta} = -\bar{a}\sin(\theta)\,\mathbf{e}_{\mathbf{x}}^{*} - \sin(\theta)\,\mathbf{e}_{\mathbf{y}}^{*} + \cos(\theta)\,\mathbf{e}_{\mathbf{z}}^{*} \tag{10}$$

whose length is given by:

$$|\frac{d\mathbf{r}}{d\theta}| = \sqrt{\bar{a}^2\sin^2\theta + \sin^2\theta + \cos^2\theta} = \sqrt{1+\bar{a}^2\sin^2\theta} \tag{11}$$

We can construct the unitary tangent vector to the boundary curve:

$$\mathbf{e_t} = \frac{\frac{d\mathbf{r}}{d\theta}}{|\frac{d\mathbf{r}}{d\theta}|} = \frac{1}{\sqrt{1+\bar{a}^2\sin^2\theta}}\{-\bar{a}\sin\theta\,\mathbf{e}_{\mathbf{x}}^{*} - \sin\theta\,\mathbf{e}_{\mathbf{y}}^{*} + \cos\theta\,\mathbf{e}_{\mathbf{z}}^{*}\} \tag{12}$$

A vector defined along the boundary curve, normal to the tangent vector $\mathbf{e_t}$ belonging to the cylindrical surface $\mathbf{e_n}$ can be produced, as depicted in Figure 2.

The triple of unit vectors defined along the boundary curve $\{\mathbf{e_n}, \mathbf{e_t}, \mathbf{e_r}\}$ can be represented in terms of the vectors of the cylindrical surface $\{\mathbf{e}_{\mathbf{x}}^{*}, \mathbf{e}_\theta, \mathbf{e_r}\}$ by:

$$\left\{\begin{array}{c}\mathbf{e_n}\\ \\ \mathbf{e_t}\\ \\ \mathbf{e_r}\end{array}\right\} = \left[\begin{array}{lll} -\frac{1}{\sqrt{1+\bar{a}^2\sin^2\theta}} & -\frac{\bar{a}}{\sqrt{1+\bar{a}^2\sin^2\theta}}\sin\theta & 0 \\ & & \\ -\frac{\bar{a}}{\sqrt{1+\bar{a}^2\sin^2\theta}}\sin\theta & \frac{1}{\sqrt{1+\bar{a}^2\sin^2\theta}} & 0 \\ & & \\ 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c}\mathbf{e}_{\mathbf{x}}^{*}\\ \\ \mathbf{e}_\theta\\ \\ \mathbf{e_r}\end{array}\right\}$$

## 4. FLUX BOUNDARY CONDITION

Let us consider again Laplace's equation on a cylindrical surface:

$$\nabla^2 u = 0 \tag{13}$$

with the boundary conditions

$$\begin{array}{llll} u_{,\mathbf{n}}(x,\theta) = & u_{,\mathbf{n}}^{*}(x,\theta) & (x,\theta) & \in\ \Gamma^1 \\ u_{,\mathbf{n}}(x,\theta) = & 0 & (x,\theta) & \in\ \Gamma^2\ , x\to\infty \end{array} \tag{14}$$

being $u_{,\mathbf{n}}(x,\theta)$ the normal derivative of the function $u(x,\theta)$ along the boundary curve.

In fact this is the same problem as the potential problem, but with a different boundary condition imposed along the boundary curve $\Gamma^1$. Consequently, $u(x,\theta)$ found using the method of separation of variables is a solution for the problem.

Let us consider a single harmonic and then latter generalize the results obtained. The gradient of the function $u = A_n \cos n\theta e^{-n\bar{x}}$ corresponds to:

$$\nabla u = -A_n n \cos n\theta e^{-n\bar{x}}\,\mathbf{e}_{\mathbf{x}^*} - A_n n \sin n\theta e^{-n\bar{x}}\,\mathbf{e}_\theta \tag{15}$$

where $\{e_{\mathbf{x}}^{*}, e_\theta\}$ are the vectors placed on the cylindrical surface.

The surface unit vectors $\{\mathbf{e}_{\mathbf{x}}^{*}, \mathbf{e}_\theta\}$ can be represented in terms of the reference vectors by:

$$\left\{\begin{array}{ll} \mathbf{e}_{\mathbf{x}^*} = & \mathbf{e_x} \\ \mathbf{e}_\theta = & -\sin\theta\,\mathbf{e_y} + \cos\theta\,\mathbf{e_z} \end{array}\right. \tag{16}$$

The gradient of the field function $u$ in terms of the reference vectors is:

$$\nabla u = A_n\{-n\cos n\theta\,\mathbf{e}^*_{\mathbf{x}} + n\sin\theta\sin n\theta\,\mathbf{e}^*_{\mathbf{y}} - n\cos\theta\sin n\theta\,\mathbf{e}^*_{\mathbf{z}}\}e^{-n\bar{x}} \tag{17}$$

Consequently, the flux across the intersection curve corresponds to the normal derivative along the direction of the normal vector to the boundary curve:

$$u_{,\mathbf{n}} = \nabla u \cdot \mathbf{e_n}$$

$$u_{,\mathbf{n}} = \frac{A_n}{\sqrt{1+\bar{a}^2\sin\theta^2}}\{n\cos n\theta + n\bar{a}\sin\theta\sin n\theta\}e^{-n\bar{x}} \tag{18}$$

Once again, expanding the exponential along the boundary curve in terms of Bessel functions:

$$e^{z\cos(\theta)} = I_0(z) + 2\sum_{k=1,2,\ldots}^{\infty} I_k(z)\cos(k\theta) \tag{19}$$

Introducing the trigonometric relation [2] into the equation describing the flow across the boundary curve, we obtain the following form:

$$\begin{aligned}
u_{,\mathbf{n}} = & \frac{1}{\sqrt{1+\bar{a}^2\sin\theta^2}}A_n\{ \\
& + \frac{1}{2}n\bar{a}\{I_0(-n\bar{a})\cos(n-1)\theta + \ldots I_k(-n\bar{a})[\cos(n-1-k)\theta + \cos(n-1+k)\theta]\} \\
& + \ n\{I_0(-na)\cos(n)\theta + \ldots I_k(-n\bar{a})[\cos(n-k)\theta + \cos(n+k)\theta]\} \\
& - \frac{1}{2}n\bar{a}\{I_0(-n\bar{a})\cos(n+1)\theta + \ldots I_k(-n\bar{a})[\cos(n+1-k)\theta + \cos(n+1+k)\theta]\}\}
\end{aligned} \tag{20}$$

We can present the result in a matrix form, which constitutes the matrix form of the harmonic-Bessel series for the prescribed flux across the elliptical boundary curve. It corresponds to:

$$\mathbf{u}_{,\mathbf{n}} = \mathbf{I}_{u,\mathbf{n}}\mathcal{A}$$

where:

$$\mathbf{u}_{,\mathbf{n}} = \left\{\begin{array}{c} u^0_{,\mathbf{n}} \\ u^1_{,\mathbf{n}} \\ \vdots \\ u^n_{,\mathbf{n}} \end{array}\right\} \text{ and } \mathcal{A} = \left\{\begin{array}{c} \mathcal{A}_0 \\ \mathcal{A}_1 \\ \vdots \\ \mathcal{A}_n \end{array}\right\}$$

The coefficient matrix relating the Fourier coefficients of flow to the Fourier coefficient of the unknowns can be represented with few terms by:

[2] Trigonometric identity:

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$
\mathbf{I}_{u,\mathbf{n}} = \begin{bmatrix} 0 & +\frac{1}{2}\bar{a}I_0(-\bar{a}) + I_1(-\bar{a}) - \frac{1}{2}\bar{a}I_2(-\bar{a})\ldots \\ 0 & \frac{1}{2}\bar{a}[I_1(-\bar{a}) + I_1(-\bar{a})] + [I_0(-\bar{a})] + \frac{1}{2}\bar{a}[I_3(-\bar{a}) + I_1(-\bar{a})]\ldots \\ \vdots & \vdots \\ \ldots & \ldots \end{bmatrix}
$$

We can express in a system of equations both boundary problems in terms of the corresponding Fourier coefficients:

$$
\begin{cases} \mathbf{u} & = \mathbf{I}_u \ \mathcal{A} \\ \mathbf{u}_{,\mathbf{n}} & = \mathbf{I}_{u,\mathbf{n}}\mathcal{A} \end{cases} \tag{21}
$$

Solving the system of equations we can find a relation between the Fourier harmonics of $u$ and $u_{,\mathbf{n}}$, as:

$$
\mathbf{u}_{,\mathbf{n}} = \mathbf{I}_{u,\mathbf{n}}\mathbf{I}_u^{-1}\mathbf{u} \tag{22}
$$

## 5. NUMERICAL COMPARISON AND CONCLUSIONS

The purpose now is to consider an alternative for the Fourier series method. For reasons concerning the elegance of the method, the boundary element method was considered. Certainly, most of those topics related to the implementation of the method are discussed extensively in the literature, so we will proceed forward to the comparison of the solutions.

A series of plots for prescribed potential along the boundary curve are analyzed. The angle of intersection ( parameter $\frac{a}{R}$) is increased in the sequence of plots. In Figure 3, the potential is prescribed along the boundary curve. The normal flux across the boundary curve produced using the Fourier series is compared to the results produced using a boundary element technique.

In Figure 4, a similar situation is analyzed by prescribing the potential along the boundary curve. The normal flux solution using Fourier series and boundary element method is presented. The solution from the boundary element method presents oscillations near the corner of the boundary curve. Certainly, additional elements are required to improve the solution.

In Figure 5, the angle of the intersection is increased and the results from boundary element method and Fourier series are compared. For this case, the Fourier series solution starts showing convergence problems around the corner of the boundary curve.

Finally, in Figure 6, the angle of the intersection has been increased once more and only the solution from the boundary element method converged. It can be observed that the boundary element and the Fourier series approach are basically equivalent. Increasing the parameter $\frac{a}{R}$, it can be observed that both approaches consistently give almost identical results. However, for values of the parameter $\frac{a}{R}$ larger then 0.6, the boundary element method is more reliable. The Fourier series approach starts having convergence problems close to the corner point ($\frac{y}{R} = \pi$).

The plane problems considered can be handled using the Fourier series and the boundary element methods. The Fourier series is faster since the problem taken for comparison requires $12$ harmonics for the analysis. For the boundary element method solution we used a total of 32 elements for the part under consideration, even though it could be handled with less elements. The boundary element comparison used quadratic boundary elements for describing the geometry and quadratic elements for the interpolation functions of the potential and flux across the boundary curve. The boundary element required more cpu time to achieve the solution, since it requires the numerical integration for producing the influence coefficients. Certainly, the cpu time can be shortened using the closed-form expressions for those coefficients as presented in the corresponding reference.

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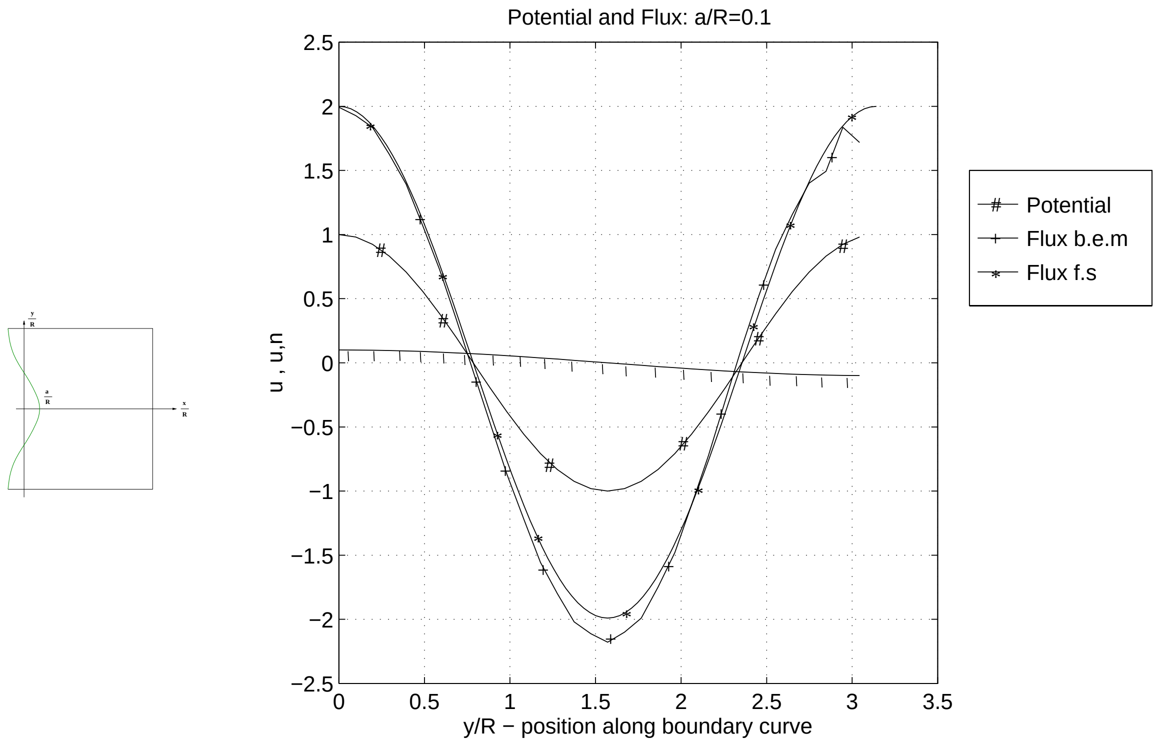


**Figure 3:** Fourier Series and Boundary Element - $\frac{a}{R} = 0.1$

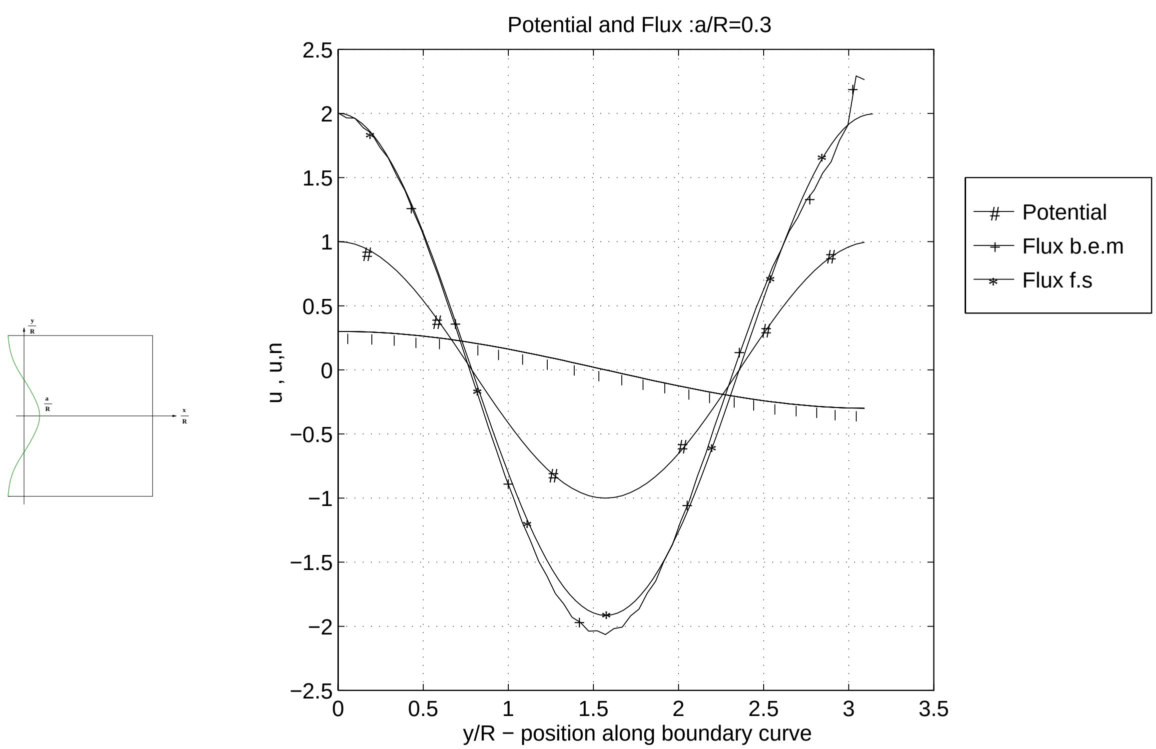


**Figure 4:** Fourier Series and Boundary Element - $\frac{a}{R} = 0.3$

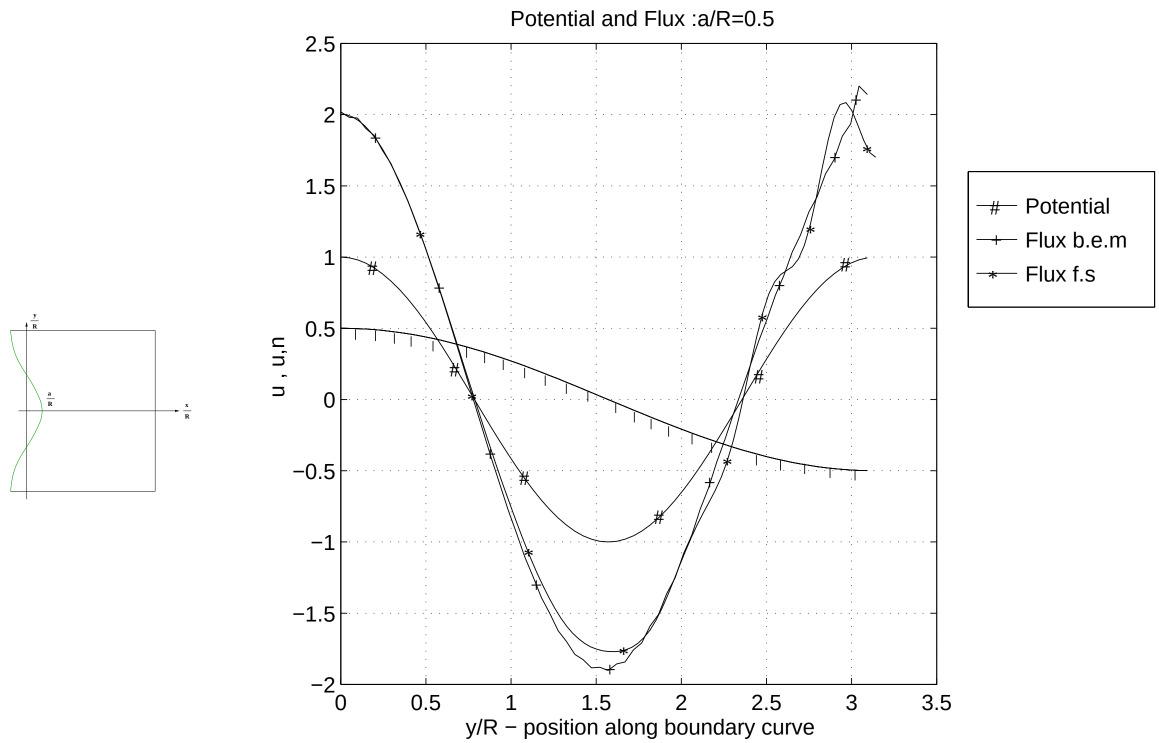


**Figure 5:** Fourier Series and Boundary Element - $\frac{a}{R} = 0.5$

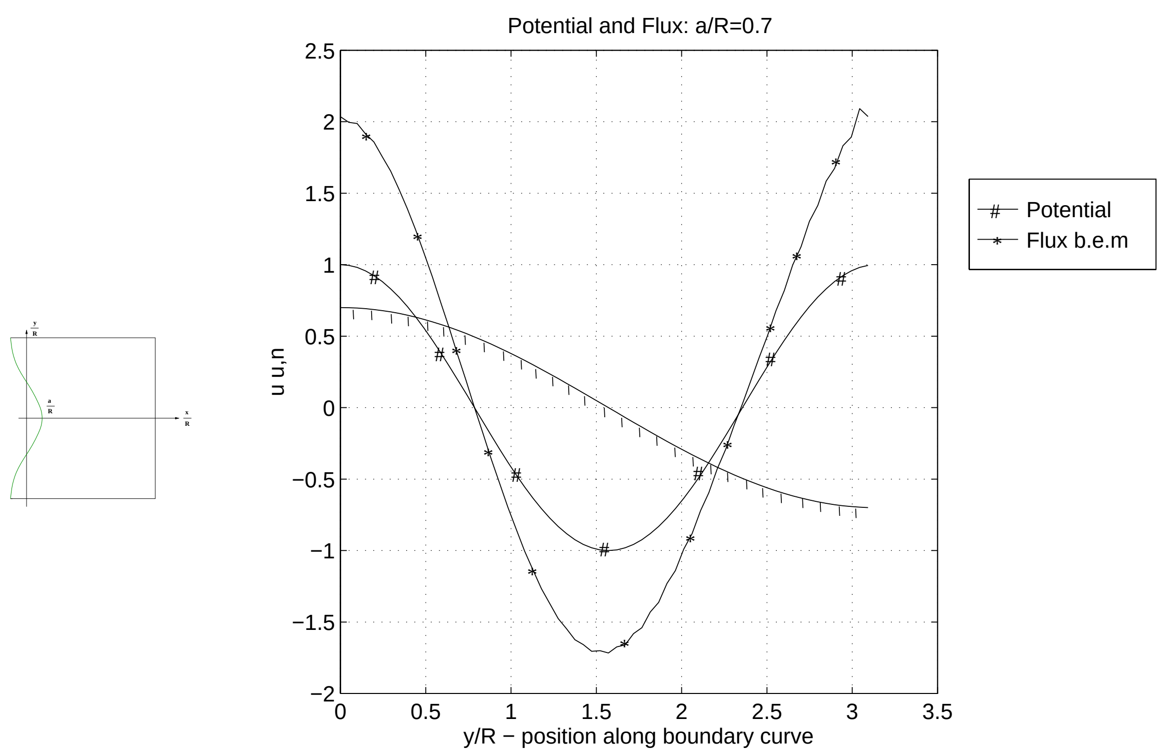


**Figure 6:** Fourier Series and Boundary Element - $\frac{a}{R} = 0.7$